Dynamic Analysis of Harmonically Excited Non-Linear Structure System Using Harmonic Balance Method

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An analytical method is presented for evaluation of the steady state periodic behavior of nonlinear structural systems. This method is based on the substructure synthesis formulation and a harmonic balance procedure, which is applied to the analysis of nonlinear responses. A complex nonlinear system is divided into substructures, of which equations are approximately transformed to modal coordinates including nonlinear term under the reasonable procedure. Then, the equations are synthesized into the overall system and the nonlinear solution for the system is obtained. Based on the harmonic balance method, the proposed procedure reduces the size of large degrees-of-freedom problem in the solving nonlinear equations. Feasibility and advantages of the proposed method are illustrated using the study of the nonlinear rotating machine system as a large mechanical structure system. Results obtained are reported to be an efficient approach with respect to nonlinear response prediction when compared with other conventional methods.

Key Words: Modeling of Non-Linear Structure, Dynamic Analysis, Modal Analysis, Harmonic Balance Method, Multi-DOF System, Design of Aircraft Gas Turbine

1. Introduction

In recent years, industrial rotating machines of industrial field trend toward the high-speed and lightweight. These conditions may cause the trouble of nonlinear vibration. In general, nonlinear vibration of rotating machinery occurs by nonlinear characteristics in the shaft or bearing. In the present rotating machinery, nonlinear vibration phenomena sometimes occur in the shrinkage fit rotor, in the assembly rotor and in the power plant rotor with coil. Nonlinear vibration phenomena also occur in a high polymer rotor which is used for lightweight construction. Vibration analysis of such rotor systems is performed usually by the linear FEM (finite element method). When large amplitude vibration occurs, however, linearized spring and damping coefficients can't model the complicated nonlinear rotor system. It is important, therefore, to consider the nonlinear characteristics in vibration analysis and design of rotor systems.

Therefore, a more accurate approach is needed to analyze the vibration of rotor system by considering its nonlinearity. In the analysis of a large complex DOF (degrees of freedom) mechanical structure, discreting the system by the FEM leads to a large number of equations of motion. Besides, the large size problem requires longer computation times. The SSM (substructure synthesis method) has been studied for efficient vibration analysis of large complex structure systems. For nonlinear structure systems, since exact solutions are generally not possible, ap-

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proximate solutions are obtained by numerical integration of nonlinear equations of motion (Dokainish and Subbarj, 1989). This method , however, may be extremely time-consuming and thus not practical especially for large DOF systems. Therefore, approximate analytical techniques (Glisinn, 1982 ; Choj and Noah, 1987 ; Haguang and Mook, 1987) are used as an alternative to numerical integration for the analysis of nonlinear systems. Iwatsubo et al. (1998 ; 1999) proposed an approximate analytical method to analyze the dynamic problems of a nonlinear Multi-DOF system using the SSM and a perturbation method. They applied the SSM technique to reduce the overall size of the problem and obtained approximate solutions by applying the perturbation method. Moon et al. (1999 ; 2001) presented an analytical method to analyze the vibration of a nonlinear mechanical systems by applying the perturbation method. They considered the nonlinearity in the shaft and bearing part and considered the effect of nonlinear sensitivity in the subsystem. They derived the formulation under the condition that the exciting force is near the first critical frequency of the system. However, a vibration problem occurs not only near the natural frequency but also in a wider frequency range in some industrial mechanical systems.

Fundamentally, there are no specific frequency range limits in analyzing the nonlinear characteristics by the harmonic balance method. Therefore, this paper presents an analytical technique based on the harmonic balance theory and the mode superposition principle for the dynamic analysis of nonlinear mechanical systems. By applying the harmonic balance method, the governing equations of the complex nonlinear system become in a compact form and can be solved in a wider frequency range than critical natural frequency of the system. Furthermore, the proposed method enhanced the former studies (Iwatsubo et al., 1998 ; 1999) such that it can be applied to wider frequency range including the dominant vibration frequency, comparing with the perturbation method of the previous studies (Iwatsubo et al., 1999). Theoretical basis of the proposed method is presented in the derivation of the fundamental harmonic response of a nonlinear structure. The proposed method is then applied to a nonlinear mechanical system in order to demonstrate the performance of the method in the respect of the computational accuracy by comparing the results with those from the other conventional methods.

2. Method of Analysis

A structural system consists of a set of interconnected components that have segments with distributed mass and elasticity and nonlinear parts. Nonlinear structures can be divided into linear and nonlinear substructures with assembling regions. The first stage in the analysis process, therefore, is the sub-structuring of the original nonlinear system into some components that can be modeled separately with linear and nonlinear sets. Small substructures may be easier to model and will eventually result in an economical analysis procedure.

2.1 Modeling of the system

In this paper, a rotor-bearing-casing system as shown in Fig. 1 is considered. The rotor is supported by bearings that are fixed on the casing. The casing and the foundation are elastically connected. The rotor has the material nonlinearity. For dynamic analysis of this kind of complex system, the SSM can be applied. The whole system is divided into three components. The rotor has nonlinear restoring force so that it is regarded as a nonlinear component while the casing is considered to be a linear component and the bearing is modeled as a linear assembling component.

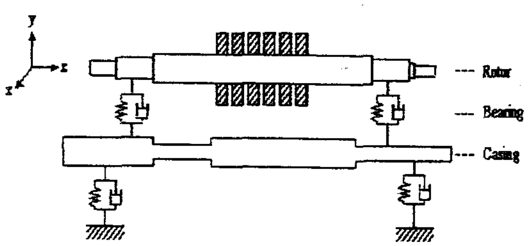


Fig. 1 Rotor-bearing-casing system

The coordinate system of the rotor-bearing-casing system is shown in Fig. 1. The $o-xyz$ coordinate system is fixed in space, where the x-axis is perpendicular to directions of shaft and casing, the y-axis is vertically upwards, and the z-axis is along the shaft and the casing for consistency of modeling. The acceleration of gravity is ignored for simplicity. The rotor gyroscopic effect caused by nonlinear restoring force is not considered in this study for simplicity of the nonlinear analysis. Instead, a proportional damping model is considered in the equations of motion. The shaft and casing components are modeled by using the FEM. A common form of excitation of a rotor system is the mass unbalance of the rotor. Then assumption of a steady state response is reasonable. The excitation forces at a given station by the unbalance mass m_i at a distance e_i from the rotor geometric center are given by

$$\{F(\Omega, t)\}\begin{Bmatrix} F_x \\ F_y \end{Bmatrix}=\begin{Bmatrix} m_i e_i \Omega^2 \cos(\Omega t+\phi_i) \\ m_i e_i \Omega^2 \sin(\Omega t+\phi_i) \end{Bmatrix}, \ (i=1, 2, 3, \cdots) \quad (1)$$

where ϕ, Ω are the phase quantity and the rotating frequency. The excitation force by the mass unbalance of the rotor can be treated as a harmonic excitation force. In general, the response shows well the nonlinear characteristics around the natural frequency in the nonlinear system, as well observed with the one DOF system. Especially in the rotor system, the dynamic behavior around the critical speed is very important where most of the troubles occur. Therefore, it is needed to pass quickly the critical speed without troubles. Because of these reasons, the exciting frequency around the 1st natural frequency of the system is considered in this study. The nonlinear restoring force is transformed into modal coordinates under the condition that the excitation force is near the first natural frequency (Moon et al., 1999 ; Moon and Kang 2001). In this study, the nonlinear term is handled with the harmonic balance method. This technique can be applied to an even lower frequency range than the first natural frequency if the first vibration mode of the system is dominant. The excitation force of the casing component is treated as general force F_p.

2.2 Modeling of nonlinear component

When the rotor is modeled by the FEM, the characteristic of nonlinear restoring force for each element is based on the relation that the stress of the element by bending moment is represented by sum of two terms, one that varies linearly with the strain plus another one that varies with the third power of the strain (Iwatsubo et al., 1998 ; 1999). Then, the nonlinear stiffness term is determined approximately in the same way as the linear stiffness term by the principle of minimum potential energy. The nonlinear restoring force is formulated in a complex form including nonlinear displacement terms. The external force, which acts on the rotor, is considered as the unbalance force and the internal force is considered because the nonlinear component can be synthesized through the internal force with the other components. By considering the boundary conditions, the equation of motion for the nonlinear component can be written as (Moon et al., 1999 ; Moon and Kang 2001)

$$[M_1]\{\ddot{u}_1\}+[K_1]\{u_1\}+\varepsilon[K_N]\{u_1^3\}=\{F(\Omega, t)\} \quad (2)$$

where $[M_1]$, $[K_1]$ are the mass and stiffness matrices, respectively. $\{F\}$ is an external force vector, $[K_N]\{u_1^3\}$ is a nonlinear term, and ε is a small parameter. The subscript 1 denotes the nonlinear component. The displacement vector can be written as

$$\{u_1\}=\{x_i, y_i, \theta_{xi}, \theta_{yi}\}^T, \ (i=1, 2, \cdots, n) \quad (3)$$

where x_i, θ_{xi} and y_i, θ_{yi} are the displacements and rotations for the x-direction and y-direction in the i-th nodal point, n is the number of nodes. Exactly to say, vibration modes of a nonlinear system are changed from those of a linear system. But for simplicity of analysis, they are assumed to keep those of a linear system. Accordingly, the modal coordinate system can be obtained by using the modal matrix $[\Phi_1]$ of the linear system. Then, the displacement $\{u_1\}$ in a physical coordinate system can be transformed into the modal coordinate $\{\xi_1\}$ system as

$$\{u_1\}=[\Phi_1]\{\xi_1\} \quad (4)$$

Substituting Eq. (4) into Eq. (2) and pre-multiplying both sides of Eq. (2) by $[\Phi_1]^T$, Eq. (2) is expanded to the nonlinear modal equation as

$$[I]\{\ddot{\xi}_1\}+[w_1^2]\{\xi_1\}+\varepsilon[\Phi_1]^T[K_N]\{u_1^3\}=\{f(\Omega, t)\} \quad (5)$$

where $\{f\}=[\Phi_1]^T\{F\}$ is the external force in modal coordinates. Usually $[\Phi_1]^T[K_N]\{u_1^3\}$ is not a diagonal matrix. This term will be changed into modal coordinates in accordance with the reasonable procedure, as shown in reference (Moon et al., 1999 ; Moon and Kang 2001). Accordingly, the nonlinear term is approximated as a diagonal matrix resulting in an efficient analysis by adopting a small number of lower frequency modes. Then, a nonlinear term can be derived as $\varepsilon[k'_m]\{\xi_1^3\}$ where $[k'_m]$ is the diagonal matrix term when the system is excited around the first natural frequency. To apply the harmonic balance method, Eq. (5) can be modified as

$$[I]\{\eta_1''\}+[I]\{\eta_1\}+\left[\frac{k'_N}{w_1^2}\right]\{\eta_1^3\}=\sqrt{\varepsilon}\left\{\frac{f(\nu_1, \tau_1)}{w_1^2}\right\} \quad (6)$$

where $\tau_1=w_1 t$, $\nu_1=\frac{\Omega}{w_1}$, $\{\eta_1\}=\sqrt{\varepsilon}\{\xi_1\}\left('=\frac{d}{d\tau_1}\right)$. Thus, the nondimensional equation of a nonlinear substructure is obtained as

$$[I]\{\eta_1''\}+\left[\frac{1}{\nu_{1n}^2}\right]\{\eta_1\}+\left[\frac{k'_N}{\Omega^2}\right]\{\eta_1^3\}=\sqrt{\varepsilon}\left\{\frac{f(\theta_1)}{\Omega^2}\right\}+\{f_{1in}\} \quad (7)$$

where $\theta_1=\nu_1\tau_1$, $\left('=\frac{d}{d\theta_1}\right)$. The internal force is considered because the nonlinear component can be synthesized through the internal force with the other components. Thus, the internal force is included in each substructure. $\{f_1\}$ is the internal force, and $\{f_{1in}\}=\sqrt{\varepsilon}\left\{\frac{f_1}{\Omega^2}\right\}$.

2.3 Modeling of linear and assembling components

The casing is modeled as a linear substructure using the FEM. After the eigenvalue analysis, the equation of motion in the modal coordinates is obtained as

$$[I]\{\ddot{\xi}_2\}+[w_2^2]\{\xi_2\}=\{f_p\} \quad (8)$$

where $[w_2^2]$ and $[I]$ are eigenvalue of linear substructure and identity matrices, respectively. $[f_p]$ is the external force vector. The internal force is introduced in the equation because the linear substructure can be assembled through the internal force with the other substructures. Thus, the nondimensional equation of the linear substructure is obtained in the same procedure as the nonlinear component as

$$[I]\{\ddot{\xi}_2\}+[w_2^2]\{\xi_2\}=\{f_p\}\sqrt{\varepsilon}\left\{\frac{f_p}{\Omega^2}\right\}-\sqrt{\varepsilon}\left\{\frac{f_1}{\Omega^2}\right\} \quad (9)$$

Ball bearings are considered as assembling components. Generally, there is a damping term in the bearing, but it is ignored in this study for the simplified model of bearing in order to verify the effect of nonlinear restoring force. The restoring force of the bearing is modeled as a linear force. In this case the force and displacement are expressed as

$$\begin{bmatrix} k_{bxx} & k_{bxy} \\ k_{byx} & k_{byy} \end{bmatrix}\begin{Bmatrix} x_b \\ y_b \end{Bmatrix}=\begin{Bmatrix} f_{bx} \\ f_{by} \end{Bmatrix} \quad (10)$$

Equation (10) can be expressed in a simpler form as

$$[K_b]\{X_b\}=\{f_b\} \quad (11)$$

where $[K_b]$ is a linear stiffness matrix, $\{X_b\}$ is the displacements in x, y directions of the rotor and casing corresponding to the bearings, and $\{f_b\}$ is the bearing force. To apply the harmonic balance method, Eq. (11) can be modified in the same way as a linear substructure. Accordingly, the nondimensional equation is obtained as

$$\frac{1}{\Omega^2}[K_b]\{\sqrt{\varepsilon}x_{c1}, \sqrt{\varepsilon}x_{c2}\}^T=\left\{-\frac{\sqrt{\varepsilon}}{\Omega^2}f_1, \frac{\sqrt{\varepsilon}}{\Omega^2}f_1\right\}^T \quad (12)$$

where $\sqrt{\varepsilon}\{x_{cl}\}=[\phi_{Acl}]\{\eta_l\}$, $(l=1, 2)$.

2.4 Application of harmonic balance method

Against a harmonic excitation, the response can be represented in terms of Fourier series. The response of the structure consists of the bias term, the fundamental harmonic term, and the superharmonic term. In this paper, the response of the structure is subjected to displacement-dependent nonlinearity. Therefore, the response of the structure consists of the fundamental harmonic term and the superharmonic terms. The terms with the odd number frequency were regarded as the main elements and the even number frequency elements, which are small for the obtained solution, were neglected. Accordingly, the solutions can be expressed using the Fourier series as

$$\{\eta_l\}=\{A_{l0}\}+\{A_{l1}\}\cos\theta+\{A_{l3}\}\cos 3\theta+\{B_{l1}\}\sin\theta +\{B_{l3}\}\sin 3\theta=\{\eta_{l0}\}+\{\eta_{l1}\}+\{\eta_{l3}\}+\{\eta_{l-1}\}+\{\eta_{l-3}\} \quad (13)$$

where A_l, B_l are the coefficients of each harmonic equation. The external force, which acts on the rotor, is considered to be the unbalance force and the internal force is considered because the nonlinear component can be synthesized through the internal force with the other components. The harmonic balance is taken by substituting Eq. (13) into Eq. (6), Eq. (9) and Eq. (12). After arranging the nonlinear term in the right side of the equation, the equations of motion for each substructure is obtained in accordance with each frequency element. For instance, equation of motion in accordance with each frequency element of linear substructure can be expressed as

$$\begin{aligned}
\cos 3\theta &: [I]\{\eta_{23}''\}+\left[\backslash\frac{1}{\nu_2^2}\backslash\right]\{\eta_{23}\} =\{f_{A23}\}-[\phi_{Ac2}]\frac{\sqrt{\varepsilon}}{\Omega^2}f_{13}, \\
\cos\theta &: [I]\{\eta_{21}''\}+\left[\backslash\frac{1}{\nu_2^2}\backslash\right]\{\eta_{21}\} =\{f_{A21}\}-[\phi_{Ac2}]\frac{\sqrt{\varepsilon}}{\Omega^2}f_{11}, \\
0 &: [I]\{\eta_{20}''\}+\left[\backslash\frac{1}{\nu_2^2}\backslash\right]\{\eta_{20}\} =\{f_{A20}\}-[\phi_{Ac2}]\frac{\sqrt{\varepsilon}}{\Omega^2}f_{10}, \\
\sin\theta &: [I]\{\eta_{2-1}''\}+\left[\backslash\frac{1}{\nu_2^2}\backslash\right]\{\eta_{2-1}\} =\{f_{A2-1}\}-[\phi_{Ac2}]\frac{\sqrt{\varepsilon}}{\Omega^2}f_{1-1}, \\
\sin 3\theta &: [I]\{\eta_{2-3}''\}+\left[\backslash\frac{1}{\nu_2^2}\backslash\right]\{\eta_{2-3}\} =\{f_{A2-3}\}-[\phi_{Ac2}]\frac{\sqrt{\varepsilon}}{\Omega^2}f_{1-3}
\end{aligned} \quad (14)$$

where $[\phi_{Ac2}]$ is the eigenvector of the assembling part in a linear substructure. Even though the casing component is a linear substrucure, harmonic balance is taken for this substructure according to the nonlinear substructure because the higher order harmonic oscillations that occur in the nonlinear substructure is translated through the higher order equation. In a similar way, equations of motion for the nonlinear substructure in accordance with each frequency element can be expressed as

$$[I]\{\ddot{\xi}_{1m}\}+\left[\backslash\frac{1}{\nu_1^2}\backslash\right]\{\eta_1\}=\{f_{A1m}\}-[\phi_{Ac1}]\frac{\sqrt{\varepsilon}}{\Omega^2}f_{1m} \quad (15)$$

where $\{f_{A1m}\}$ is a function of A_l, B_l which are the coefficient components in each harmonic equation (Eq. (13)), $[\phi_{Ac1}]$ is an eigenvector of the assembling region of the nonlinear substructure ($m=3,\ 1,\ 0,\ -1,\ -3$). The linear assembling region is also formulated by Fourier series like the linear substructure. The equation of assembling region is obtained for corresponding frequency after taking the harmonic balance by considering the assembling condition and Eq. (13) as

$$\frac{1}{\Omega^2}[K_b]\{\sqrt{\varepsilon}x_{c1m},\ \sqrt{\varepsilon}x_{c2m}\}^T=\left\{-\frac{\sqrt{\varepsilon}}{\Omega^2}f_{1m},\ \frac{\sqrt{\varepsilon}}{\Omega^2}f_{1m}\right\}^T \quad (16)$$

By assembling each substructure with its truncated vibration modes according to the SSM procedure, the reduced order equation of motion for the overall structure can be obtained. By synthesizing Eq. (14), Eq. (15), and Eq. (16), the equation for the overall structure is obtained as

$$[M_z]\{\eta_z''\}+[K_z]\{\eta_z\}=\{f_z\} \quad (17)$$

where $[M_z]$, $[K_z]$ and $\{f_z\}$ are the mass matrix, stiffness matrix and external force of the overall structure, respectively. Those terms are composed of corresponding substructure term and assembling region term. The coordinate vector of the overall structure is $\{\eta_z\}=\{\eta_{1m},\ \sqrt{\varepsilon}x_{c1m},\ \sqrt{\varepsilon}x_{c2m},\ \eta_{1m}\}^T$. In order to apply the SSM, the following coordinate transform matrix $[T_p]^T$ is introduced as

$$\{\eta_z\}=[I,\ \phi_{Ac1},\ \phi_{Ac2},\ I]^T\{\eta_{1m},\ \eta 2_{1m}\}^T=[T_p]\{\bar{\eta}\} \quad (18)$$

where I stands for a identity matrix. Substituting Eq. (18) into Eq. (17) and pre-multiplying from the left with $[T_p]^T$ yields

$$[M']\{\bar{\eta}_z''\}+[K']\{\bar{\eta}_z\}=\{f_z'\} \quad (19)$$

where $[M']$, $[K']$ are the reduced order mass matrix and stiffness matrix terms, respectively,

$\{ f'_z \}$ is the external force term. Equation (19) is independent in each frequency element. Thus, the assembled overall equation for each frequency element can be expressed as

$$[M']\{ \bar{\eta}''_m \}+[K]\{ \eta_m \}=-\{\{ f_{A1m} \}, 0 \}^T \quad (20)$$

$$[M]=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \{ \eta_m \}=\begin{Bmatrix} \eta_{1m} \\ \eta_{2m} \end{Bmatrix},$$

$$[K]=\begin{bmatrix} \frac{1}{\nu_1{}^2}+\frac{1}{\Omega^2}\phi^T_{Ac1}K_b\phi_{Ac1} & -\frac{1}{\Omega^2}\phi^T_{Ac1}K_b\phi_{Ac2} \\ -\frac{1}{\Omega^2}\phi^T_{Ac2}K_b\phi_{Ac1} & \frac{1}{\nu_2{}^2}+\frac{1}{\Omega^2}\phi^T_{Ac2}K_b\phi_{Ac2} \end{bmatrix}$$

Generally, there is an external force in each substructure. In this study, a rotor-bearing-casing system is considered as an analytical model. Accordingly, the external force for the casing can be neglected. By applying the modal analysis technique, Eq. (20) can be solved for corresponding frequency elements using the relation of $\{ \eta_m \}=[\Phi_z]\{ \zeta_m \}$ where $[\Phi_z]$ is the modal matrix of the overall structure. The equation of the overall structure can be expressed as for each frequency element as

$$[I]\{ \zeta''_m \}+[w_z^2]\{ \zeta_m \}=[\Phi_z]\{ f_{A1m} \} \quad (21)$$

where w_z^2 is the eigenvalue of the overall structure. Generally, there is a damping term in the equation of each substructure, which can be expressed by proportional damping. However in this study, damping term is considered in the overall system by a proportional damping that consists of the mass and stiffness term of the overall system as $[C]=\alpha[I]+\beta[w_z^2]$. The aim of this study is to verify the nonlinear characteristic of the system, which has nonlinear restoring force by an analytical method. Thus, the damping term is considered in the overall structure for simplicity of nonlinear analysis, which shows the no difference in the response. The overall structure is analyzed by solving Eq. (21) around the first natural frequency of the system. The amplitude of rotor is large around the first natural frequency of the system so that the nonlinearity has much influence on the vibration of the rotor. Accordingly, the development of the accurate analytical method of nonlinear dynamic response near the 1st natural frequency is strongly desired. By considering the damping term and by taking the harmonic balance, then the equations of overall system are obtained as

$$\begin{aligned}
\cos 3\theta:\ & (-9+w_{z1}^2)\bar{\zeta}_3+3C_1\bar{\zeta}_{-3}=-P\Big\{\frac{1}{4}\bar{\zeta}_1^3+3\bar{\zeta}_0^2\bar{\zeta}_3 \\
& +\frac{3}{2}\bar{\zeta}_1^2\bar{\zeta}_3+\frac{3}{4}\bar{\zeta}_3^3-\frac{3}{4}\bar{\zeta}_1\bar{\zeta}_{-1}^2+\frac{3}{2}\bar{\zeta}_3\bar{\zeta}_{-1}^2+\frac{3}{4}\bar{\zeta}_3\bar{\zeta}_{-3}^2\Big\} \\
\cos \theta:\ & (-1+w_{z1}^2)\bar{\zeta}_1+C_1\bar{\zeta}_{-1}=f_{z1}-3P_c\{\bar{\zeta}_0^2\bar{\zeta}_1+\frac{1}{4}\bar{\zeta}_1^3+\frac{1}{4}\bar{\zeta}_1^2\bar{\zeta}_3 \\
& +\frac{1}{2}\bar{\zeta}_1\bar{\zeta}_3^2+\frac{1}{4}\bar{\zeta}_1\bar{\zeta}_{-1}^2-\frac{1}{4}\bar{\zeta}_3\bar{\zeta}_{-1}^2+\frac{1}{2}\bar{\zeta}_1\bar{\zeta}_{-1}\bar{\zeta}_{-3}+\frac{1}{2}\bar{\zeta}_1\bar{\zeta}_{-3}^2\} \\
0:\ & w_{z1}^0\bar{\zeta}_0=P_c\{\bar{\zeta}_0^3+\frac{3}{2}\bar{\zeta}_0\bar{\zeta}_1^2+\frac{3}{2}\bar{\zeta}_0\bar{\zeta}_3^2+\frac{3}{2}\bar{\zeta}_0\bar{\zeta}_{-1}^2+\frac{3}{2}\bar{\zeta}_0\bar{\zeta}_{-3}^2\} \\
\sin \theta:\ & (-1+w_{z1}^2)\bar{\zeta}_{-1}-C_1\bar{\zeta}_1=-3P_c\Big\{\bar{\zeta}_0^2\bar{\zeta}_{-1}+\frac{1}{4}\bar{\zeta}_1^2\bar{\zeta}_{-1}-\frac{1}{2}\bar{\zeta}_1\bar{\zeta}_3\bar{\zeta}_{-1} \\
& +\frac{1}{2}\bar{\zeta}_3^2\bar{\zeta}_{-1}+\frac{1}{4}\bar{\zeta}_{-1}^3+\frac{1}{4}\bar{\zeta}_1^2\bar{\zeta}_{-3}-\frac{1}{4}\bar{\zeta}_{-1}^2\bar{\zeta}_{-3}+\frac{1}{2}\bar{\zeta}_{-1}\bar{\zeta}_{-3}^2\Big\} \\
\sin 3\theta:\ & (-9+w_{z1}^2)\bar{\zeta}_{-3}-3C_1\bar{\zeta}_3=P_c\Big\{\frac{3}{4}\bar{\zeta}_1^2\bar{\zeta}_{-1}-\frac{1}{4}\bar{\zeta}_{-1}^3+3\bar{\zeta}_0^2\bar{\zeta}_{-3} \\
& +\frac{3}{2}\bar{\zeta}_1^2\bar{\zeta}_{-3}+\frac{3}{4}\bar{\zeta}_3^2\bar{\zeta}_{-3}+\frac{3}{2}\bar{\zeta}_{-1}^2\bar{\zeta}_{-3}+\frac{3}{4}\bar{\zeta}_{-3}^3\Big\} \qquad (22)
\end{aligned}$$

where C_1 is a coefficient of harmonic equation, f_{z1} is an external force term in overall system, and $f_{z1}=\frac{\sqrt{\varepsilon}}{w_{z1}^2}\sum_{i=1}^{n}\Phi_{z1t}f_t$, $P_c=\sum_{i=1}^{n}\Phi_{z1t}\frac{k'_N}{\Omega^2}\Omega_{z1t}^{T3}$. Then, the displacement coordinate $\{ \zeta \}$ can be obtained as

$$\begin{aligned}
\{ \zeta \}&=\{ \zeta_0 \}+\{ \zeta_1 \}+\{ \zeta_3 \}+\{ \zeta_{-1} \}+\{ \zeta_{-3} \} \\
&=\{ \bar{\zeta}_0 \}+\{ \bar{\zeta}_1 \}\cos \theta+\{ \bar{\zeta}_3 \}\cos 3\theta+\{ \bar{\zeta}_{-1} \}\sin \theta \\
&\quad +\{ \bar{\zeta}_{-3} \}\sin 3\theta \qquad (23)
\end{aligned}$$

For instance, a coordinate $\{ \zeta \}$ for the first mode element can be expressed as

$$\begin{aligned}
\zeta_1&=\zeta_{10}+\zeta_{11}+\zeta_{13}+\zeta_{1-1}+\zeta_{1-3} \\
&=\bar{\zeta}_{10}+\bar{\zeta}_{11}\cos \theta+\bar{\zeta}_{13}\cos 3\theta \\
&\quad +\bar{\zeta}_{1-1}\sin \theta+\bar{\zeta}_{1-3}\sin 3\theta \qquad (24)
\end{aligned}$$

The nonlinear frequency responses are obtained by solving Eq. (22). By substituting the solution of Eq. (22) into Eq. (23) and by changing the modal coordinates into physical coordinates, the time domain response can be obtained.

3. Numerical Examples

In this section, response analysis is presented to demonstrate the application of the proposed method. The frequency response is obtained by

Table 1 Properties of the rotor system

Rotor	Length (*mm*)	800
	Diameter (*mm*)	30
	Young's modulus (N/m^2)	2.1×10^{11}
	Density (kg/m^3)	7.81×10^3
	Small parameter ε	0.1
Casing	Length (*mm*)	800
	Diameter (*mm*)	100
	Young's modulus (N/m^2)	2.1×10^{11}
	Density (kg/m^3)	7.81×10^3
Bearing	Coefficient k_{bxx}, k_{byy} (N/m)	6.69×10^6

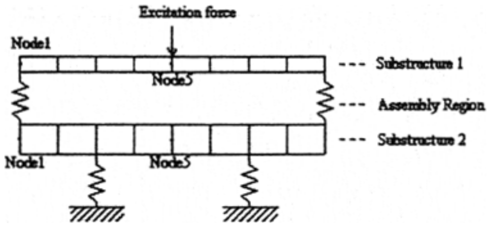


Fig. 2 Model for analysis

sweeping up the frequency gradually. The responses of the proposed method are compared with those obtained by the classical analysis technique for accuracy validation.

A nonlinear rotor system shown in Fig. 2 is considered. The rotor is considered to be a uniform beam and the casing is also considered to be a uniform beam approximately for simplicity of calculation. Generally, there are cross-coupling terms in the ball bearing, which are ignored for the simplified model of bearing to verify the effect of nonlinearity. The properties of the rotor system are tabulated in Table 1.

The rotor and casing are modeled by the eight beam finite elements. The modal damping ratios of the rotor and casing are given by α, β=0.05. The unbalance of the external force is given by a value of 50 on the 5th nodal point of substructure 1. Figure 3 shows the frequency responses at the nodal point 5 of substructure 1 when adopting 5 modes and all modes (18 modes) by solving Eq. (22). It can be observed that the presented method can simulate relatively accurate frequency responses by adopting 5 modes compared with the responses of adopting all modes. From this result, it is believed that the nonlinear restoring force

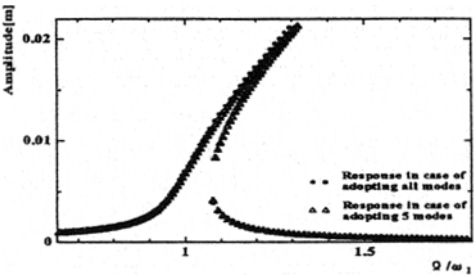


Fig. 3 Effect of adopting modes in frequency response

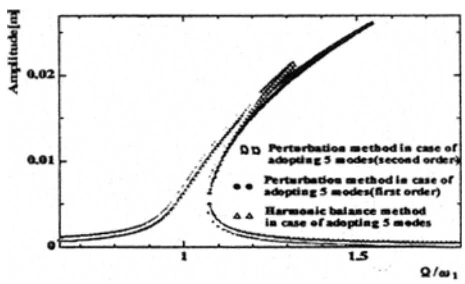


Fig. 4 Comparison of frequency responses

term can be easily transformed into modal coordinates while retaining its accuracy with its lower modes according to the proposed procedure.

To evaluate effectiveness of the proposed technique, the responses need to be compared with the other representative nonlinear analyzing method, such as perturbation method. Figure 4 shows the frequency responses, which are analyzed using the multiple scales method with the first order and the second order, as introduced in the former study (Iwatsubo et al., 1999). Those responses are obtained at the nodal point 5 of substructure 1. Five modes are adopted for each methods. As shown in reference (Hassan, 1994), "Incorrect solutions", which do not exist in the direct numerical integration response, appears to be the solution when using the method of multiple scales with approximating to the second order. Though "Incorrect solutions" appear in the large amplitude area around 0.03 or more of the frequency response curve of Fig. 4, they are neglected because it is unrelated in substance with this study. It can be observed from Fig. 4, that the results of each method simulated well the nonlinear characteristic in comparatively good accuracy. Especially, the response of the perturbation second order in accordance with the multiple scales method and the response of the harmonic balance method are in good agreements. From this result, it can be concluded that an analytical result obtained by the harmonic balance method can secure the simulation accuracy with relatively compact formulation of the complex system where the perturbation theory with the second order in accordance with the multiple scales requires a more complex formulation.

Figure 5 shows the frequency response results of the harmonic balance method and direct integration method at the node 5 of substructure 1. The response of the harmonic balance method adopted also 5 modes. It can be observed that the results of the harmonic balance method are in comparatively good agreement with the results of direct integration method even though there is a little difference in large amplitude area. As a result, the proposed method can analyze the frequency domain response with the almost same degree of accuracy with the direct numerical integration method. In this study, the steady state response is analyzed and the stability distinction is not carried out.

Next, the time domain response is described. Figures 6 and 7 show time domain response results using the harmonic balance method adopting 5 modes in accordance with Eq. (24) and the direct integration method at the node 1 and node 5 of substructure 1.

Figure 6 compares time domain responses when the system is excited by the external force with exciting frequency of 138 *rad/sec* where the first natural frequency of the system is 140 *rad/sec*.

Figure 7 compares time domain responses when the external force excites the system with exciting frequency 158*rad*/sec, which is a little larger than the first natural frequency of the system (140 *rad*/sec). Compared with the amplitude of the response by the direct integration method, it can be observed at the selected point that comparatively accurate nonlinear responses of the system are simulated with the

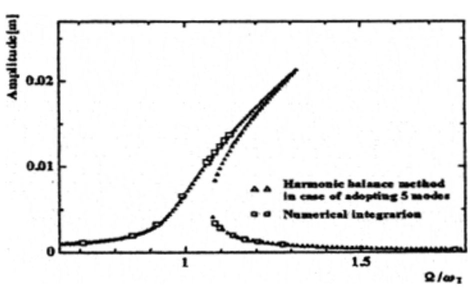
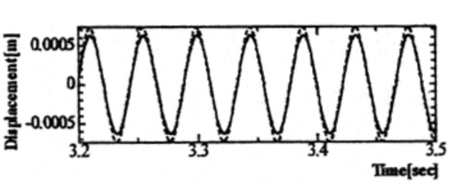
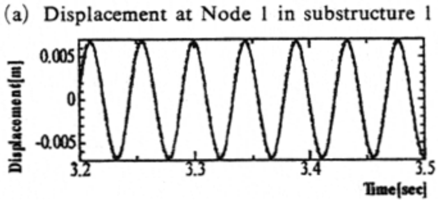


Fig. 5 Comparison of frequency responses



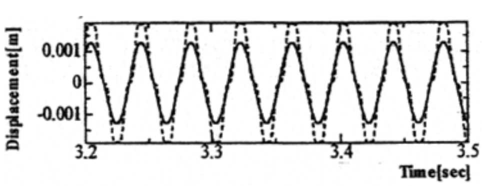
(a) Displacement at Node 1 in substructure 1



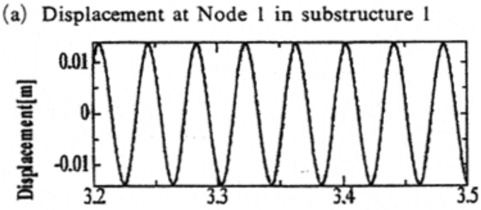
(b) Displacement at Node 5 in substructure 1

—— Presented method, ······ Integration method

Fig. 6 Comparison of time history($\Omega=138$*rad/sec*)



(a) Displacement at Node 1 in substructure 1



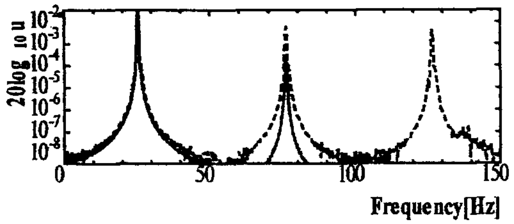
(b) Displacement at Node 5 in substructure 1

—— Presented method, ······ Integration method

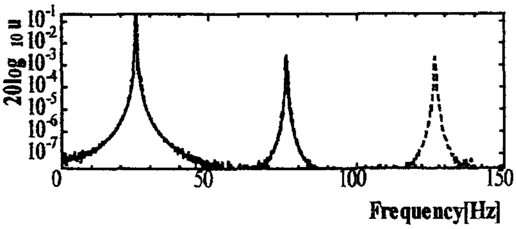
Fig. 7 Comparison of time history($\Omega=158$*rad*/sec) corresponding phase. Nevertheless, there is a little difference of responses at node 1 of substructure 1, as shown in Fig. 6, Fig. 7. Careful examination of the responses both at nodal points 1 and 5 reveals that the nonlinear displacements become small even though it is excited near the first natural frequency, as shown in Fig. 6 compared with Fig. 7. This can be understood that the amplitude of the frequency response grows up even though the exciting frequency exceeds the first natural frequency because of the effect of nonlinearity, as can be observed in Fig. 5.

Figure 8 shows the corresponding FFT(Fast Fourier Transform) analysis results of time response at nodes 1 and 5 of substructure 1, which are calculated by the direct numerical integration and the harmonic balance method. Each time response is obtained by the same simulation condition as in Fig. 7. The power spectrum is expressed in a logarithm display to confirm the nonlinear frequency element easily in the diagram. Investigation of both results reveals comparatively good agreement.

The nonlinear frequency element (3Ω) is observed in each spectrum. Nevertheless, it is observed that the spectrum of nonlinear frequency element (3Ω) of the proposed method is smaller than the spectrum obtained by the direct numerical integration method at node 1 of substructure 1. It is considered that the response of harmonic balance method is an approximated one, which may result in a difference compared with exact solution. There is no higher nonlinear frequency element (5Ω) in the proposed method where the result of the integration method shows one. Because the proposed method gives an approximation on the solution by the frequency (3Ω) element in accordance with Fourier series, there is no frequency element (5Ω).



(a) Node 1 in Substructure 1



(b) Node 5 in Substructure 1

—— Presented method, ······ Integration method

Fig. 8 Comparison of Frequency Spectra

Next, the computation time is compared to show the effectiveness of the proposed method. As an example case, the calculation time for the responses of Fig. 7 is examined. The proposed method takes 3 minutes and 56 seconds for the 3.5 second time interval, while the direct integration method takes 21 minutes and 47 seconds by using a personal computer *Logix IBM Co.*. This shows that the proposed method reduces computational time drastically while retaining the accuracy. This is a critical factor in the analysis of the structural dynamics with a large number of DOF systems.

4. Conclusions

In this paper, a vibration analysis of a nonlinear structural system is theoretically formulated applying the harmonic balance method, which can be applied to wider frequency range including the dominant vibration frequency. The formulation is concerned with reducing the number of DOF for each substructure by modal substitution in accordance with harmonic balance theory. All the substructures are then reassembled together and the nonlinear response of the overall system is obtained for the harmonic excitation. This method is applied to a nonlinear mechanical system. The performance of the proposed method is compared in terms of computational accuracy and time with the perturbation method and the direct integral method. It is shown that nonlinear responses can be efficiently calculated according to the selected number of vibration modes. And the nonlinear characteristic of the nonlinear restoring force is obtained. As a result, the proposed method can be applied in analyzing dynamics of nonlinear structures.

Acknowledgments

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